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The Extended Fréchet Distribution: Properties and Applications

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The Extended Fréchet Distribution: Properties and Applications

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This paper introduces a new four-parameter lifetime model called the Marshall-Olkin generalized Fréchet (MOGFr) distribution. We derive some of its mathematical properties including quantile and generating functions, ordinary and incomplete moments, mean residual lifetime and mean waiting time and order statistics. The MOGFr density can be expressed as a linear mixture of Fréchet densities. The maximum likelihood method is used to estimate the MOGFr parameters. Using two applications, we illustrate the flexibility and importance of the MOGFr distribution in modeling various types of lifetime data.

Keywords: Fréchet distribution, Maximum likelihood, Mean residual life, Moments, Order statistics.

1. Introduction

The Fréchet distribution (Fréchet, 1942), also known as type II extreme value distribution, has many applications in extreme value theory as an important distribution in extreme value theory.The Fréchet distribution has applications in stochastic phenomena such as rainfall, floods, air pollution (see, Kotz and Nadarajah, 2000). Further, Harlow (2002) and Zaharim et al. (2009) applied Fréchet distribution in engineering applications and in analyzing wind speed data, respectively. To explore more informations about applications of Fréchet distribution see, e.g. Kotz and Nadarajah (2000) and Resnick (2013).

Recently, the statisticians have proposed various extensions of the Fréchet distribution to increase its flexibility. For example, the exponentiated Fréchet (Nadarajah and Kotz, 2003), Beta Fréchet (Nadarajah and Gupta, 2004), Marshall-Olkin Fréchet (Krishna et al., 2013), transmuted exponentiated Fréchet (Elbatal et al., 2014), transmuted Marshall-Olkin Fréchet (Afify et al., 2015), Kumaraswamy Marshall-Olkin Fréchet (Afify et al. 2016a), Weibull Fréchet (Afify et al., 2016b), beta exponential Fréchet (Mead et al., 2017) and Burr X Fréchet (Abouelmagd et al., 2018) distributions, among others.

The survival function (SF) and probability density function (PDF) of the Fréchet (Fr) distribution are given (for $x > 0$) by

$$\overline{G}(x;\alpha,\beta) = 1 - \exp\left(-\alpha^{\beta}x^{-\beta}\right) \text{ and } g(x;\alpha,\beta) = \beta\alpha^{\beta}x^{-\beta-1}\exp\left(-\alpha^{\beta}x^{-\beta}\right), \quad (1)$$

where $\overline{G}(x;\alpha,\beta) = 1 - G(x;\alpha,\beta)$ and $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter.

Our aim in this paper is to propose and study another extension of the Fréchet model called the Marshall-Olkin generalized Fréchet (MOGFr) distribution. Its main feature is that two additional positive shape parameters are inserted in Equation (1) to provide great flexibility for the generated model. Based on the Marshall-Olkin generalized-G (MOG-G) family (Yousof et al., 2018), we construct the four-parameter MOGFr distribution and provide some of its mathematical properties. We prove that the MOGFr distribution is capable of modelling various shapes of data using two data sets. It can provide better fits than other nested and non-nested model in two real data applications.

The remainder of this article is outlined as follows: In Section2, we define the MOGFr distribution and provide its special cases. We derive some mathematical properties of the MOGFr distribution including linear representation for its PDF, quantile and generating functions, ordinary and incomplete moments, mean residual life, mean waiting time and order statistics in Section 3. The maximum likelihood estimation method is discussed in Section 4. In Section 5, the MOGFr distribution is applied to two real data sets to illustrate its importance. Finally, in Section 6, we give some concluding remarks.

2. The MOGFr distribution

Consider the SF of a baseline model, $\overline{G}(x;\boldsymbol{\xi}) = 1 - G(x;\boldsymbol{\xi})$, with a parameter vector $\boldsymbol{\xi}$, then the cumulative distribution function (CDF) MOG-G family is defined by

$$F(x;\delta,\eta,\boldsymbol{\xi}) = \frac{1-\overline{G}(x,\boldsymbol{\xi})^{\eta}}{1-(1-\delta)\overline{G}(x,\boldsymbol{\xi})^{\eta}}, x \in \mathbb{R}. \quad (2)$$

where are two positive shape parameters representing the different patterns of the MOG-G family.

The corresponding PDF and hazard rate function (HRF) of (2) are

$$f(x;\delta,\eta,\boldsymbol{\xi}) = \frac{\delta\eta g(x;\boldsymbol{\xi})\overline{G}(x,\boldsymbol{\xi})^{\eta-1}}{\left[1-(1-\delta)\overline{G}(x,\boldsymbol{\xi})^{\eta}\right]^{2}}, x \in \mathbb{R} \quad (3)$$

and

$$\tau(x;\delta,\eta,\boldsymbol{\xi}) = \frac{\eta\pi(x;\boldsymbol{\xi})}{1-(1-\delta)\overline{G}(x,\boldsymbol{\xi})^{\eta}}, x \in \mathbb{R},$$

where $g(x;\boldsymbol{\xi})$ is the PDF of a baseline model, $\pi(x;\boldsymbol{\xi}) = g(x;\boldsymbol{\xi})/\overline{G}(x,\boldsymbol{\xi})$ is the baseline HRF, δ and η are positive shape parameters. The random variable X with PDF (3) is denoted by X ~MOG-G($\delta,\eta,\boldsymbol{\xi}$). For $\eta = 1$, we have the Marshall-Olkin-G family (Marshall and Olkin, 1997), for $\delta = 1$, the MOG-G family reduces to the generalized-G family (Gupta et al., 1998), and for $\delta = \eta = 1$, we obtain the baseline distribution.

Combining (1) and (2), we obtain the CDF of the MOGFr distribution

$$F(x;\alpha,\beta,\delta,\eta) = \frac{1-\left[1-\exp\left(-\alpha^{\beta}x^{-\beta}\right)\right]^{\eta}}{1-(1-\delta)\left[1-\exp\left(-\alpha^{\beta}x^{-\beta}\right)\right]^{\eta}}, x > 0. \quad (4)$$

The PDF amd HRF of the MOGFr distribution are, respectively, given by

$$f(x;\alpha,\beta,\delta,\eta) = \frac{\delta\eta\beta\alpha^{\beta}x^{-\beta-1}\exp\left(-\alpha^{\beta}x^{-\beta}\right)\left[1-\exp\left(-\alpha^{\beta}x^{-\beta}\right)\right]^{\eta-1}}{\left\{1-(1-\delta)\left[1-\exp\left(-\alpha^{\beta}x^{-\beta}\right)\right]^{\eta}\right\}^{2}}, x > 0 \quad (5)$$

and

$$\tau(x;\alpha,\beta,\delta,\eta) = \frac{\eta\beta\alpha^{\beta}x^{-\beta-1}\exp\left(-\alpha^{\beta}x^{-\beta}\right)}{\left[1-\exp\left(-\alpha^{\beta}x^{-\beta}\right)\right]-(1-\delta)\left[1-\exp\left(-\alpha^{\beta}x^{-\beta}\right)\right]^{\eta+1}}, x > 0,$$

where α,β,δ and η are positive shape parameters. The random variable X having PDF (5) is denoted by X ~MOGFr(α,β,δ,η). The MOGFr distribution contains eleven special cases which are listed in Table 1.

Table 1: Special cases of the MOGFr distribution

α	β	δ	η	Distribution
α	β	1	η	Exponentiated Fr (Nadarajah and Kotz, 2003)
α	β	δ	1	Marshall-Olkin Fr (Krishna et al., 2013)
α	1	δ	η	MOG-inverse exponential (MOGIEx)
α	2	δ	η	MOG-inverse Rayleigh (MOGIR)
α	1	1	η	Exponentiated IEx (EIEx)
α	2	1	η	Exponentiated IR (EIR)
α	1	δ	1	Marshall-Olkin IEx (MOIEx)
α	2	δ	1	Marshall-Olkin IR (MOIR)
α	β	1	1	Fr
α	1	1	1	IEx
α	2	1	1	IR

Some plots of the PDF and HRF of the MOGFr distribution for some parameter values are displayed in Figure 1. The plots show that the PDF of the MOGFr model can be left skewed or right skewed and the MOGFr HRF can be decreasing, increasing and upside down bathtub.

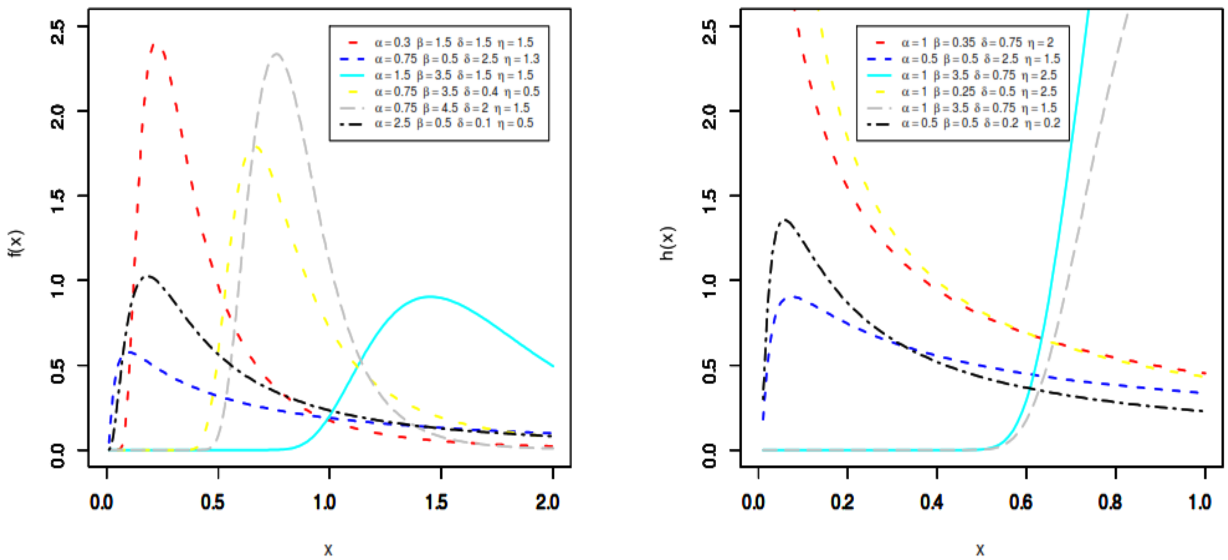

Figure 1: Plots of the PDF and HRF of the MOGFr distribution

3. Properties of the MOGFr distribution

In this section, we derive some mathematical properties of the MOGFr distribution including linear representation for its PDF, ordinary and incomplete moments, mean residual life, mean waiting time, quantile and moment generating functions and order statistics.

3.1 Linear representation

Yousof et al. (2018) derived a useful linear representation of the CDF of the MOG-G family as

$$F(x) = \sum_{k=0}^{\infty} d_k \; G(x)^k,$$

where $d_0 = (2/\delta)$ and for $k \geq 1$, we have

$$d_k = \frac{1}{\delta}\left(a_k - \frac{1}{\delta}\sum_{r=1}^{k} b_r d_{k-r}\right),$$

where $a_k = (-1)^{k+1}\binom{\eta}{k}$ and $b_r = (1-\delta)(-1)^{r+1}\binom{\eta}{r}$.

Then, the PDF of the MOG-G family can also be expressed as

$$f(x) = \sum_{k=0}^{\infty} d_{k+1} \, h_{k+1}(x),$$

where $h_{k+1}(x)$ denotes the exp-G density with positive power parameter k. Hence, the MOGFr density can be rewritten as

$$f(x) = \sum_{k=0}^{\infty} d_{k+1} (k+1)\beta\alpha^{\beta} x^{-\beta-1} \exp\left[-(k+1)\alpha^{\beta} x^{-\beta}\right].$$

Then, the PDF of the MOGFr reduces to

$$f(x) = \sum_{k=0}^{\infty} d_{k+1} g_{\alpha(k+1)}(x), \tag{6}$$

where $g_{\alpha(k+1)}(x)$ is the Fr PDF with shape parameter β and scale parameter $\alpha(k+1)^{1/\beta}$. Equation (6) reveals that many properties of the MOGFr distribution can be derived from the Fr properties.

Let Y be a random variable with Fr distribution (1) with parameters $\alpha > 0$ and $\beta > 0$. The sth ordinary and incomplete moments of Y are given (for $s < \beta$) by

$$\mu'_{s,Y} = \alpha^{s}\, \Gamma\left(1 - \frac{s}{\beta}\right) \quad \text{and} \quad \varphi_{s,Y}(t) = \alpha^{s} \gamma\left(1 - \frac{s}{\beta}, \left(\frac{\alpha}{t}\right)^{\beta}\right),$$

respectively, where $\Gamma(b) = \int_0^{\infty} z^{b-1}\, e^{-z} dz$ is the complete gamma function and $\gamma(b,t) = \int_0^{t} z^{b-1}\, e^{-z} dz$ iis the lower incomplete gamma function.

3.2 Ordinary and incomplete moments

The nth ordinary moment of X is given by

$$\mu'_n = E(X^n) = \sum_{k=0}^{\infty} d_{k+1} \int_{-\infty}^{\infty} x^n\, g_{\alpha(k+1)}(x) dx.$$

Then, we obtain

$$\mu'_r = \sum_{k=0}^{\infty} d_{k+1}\, \alpha^{n} (k+1)^{n/\beta}\, \Gamma\left(1 - \frac{n}{\beta}\right), n < \beta. \tag{7}$$

The mean of X follows by setting $n = 1$ in (7).

In Table 2 we provide numerical values for the mean, variance, skewness and kurtosis of the MOGFr distribution, for some selected parameter values of β, δ and η with $\alpha = 1$, to illustrate their effects on these measures. Table 2 shows that, for fixed δ and η, the mean, variance, skewness and kurtosis are decreasing functions of β. For fixed η and β, the mean and variance are increasing functions of δ, whereas the skewness and kurtosis are decreasing functions of δ. Further, for fixed δ and β, the mean, variance, skewness and kurtosis are decreasing functions of η. One can see, from Table 2, that the MOGFr distribution can be left skewed or right skewed. Further, it can be platykurtic (kurtosis < 3) or leptokurtic (kurtosis > 3). Hence, the MOGFr model is a flexible distribution and can be used in modeling skewed data.

Table 2: Mean, variance, skewness and kurtosis of the MOGFr distribution ($\alpha = 1$)

δ	η	β	Mean	Variance	Skewness	Kurtosis
1.5	2.5	2.0	1.0196	0.2048	2.8718	32.7633
		3.5	0.9917	0.0530	1.4555	8.1189
		5.0	0.9890	0.0245	1.1092	5.8661
		25	0.9959	0.0009	0.6011	3.8548
2.5	2.5	2.0	1.1414	0.2606	2.7173	30.5582
		3.5	1.0570	0.0622	1.3299	7.5596
		5.0	1.0339	0.0279	0.9860	5.4853
		25	1.0047	0.0010	0.4753	3.6853
5.0	2.5	2.0	1.333	0.3549	2.5632	28.6521
		3.5	1.1547	0.0756	1.1945	7.0787
		5.0	1.0998	0.0324	0.8489	5.1719
		25	1.0171	0.0011	0.3267	3.5893
1.5	3.5	2.0	0.8827	0.0948	1.7795	11.0211
		3.5	0.9190	0.0304	1.0326	5.4945
		5.0	0.9392	0.0151	0.8006	4.5172
		25	0.9862	0.0006	0.4237	3.4871
2.5	3.5	2.0	0.9691	0.1141	1.6447	10.2284
		3.5	0.9692	0.0343	0.9102	5.1575
		5.0	0.9747	0.0166	0.6789	4.2787
		25	0.9935	0.0007	0.2989	3.3911
5.0	3.5	2.0	1.1003	0.1435	1.5032	9.5439
		3.5	1.0422	0.0394	0.7741	4.8936
		5.0	1.0255	0.0184	0.5403	4.1144
		25	1.0037	0.0006	0.1499	1.3065
1.5	5.0	2.0	0.7815	0.0494	1.2201	6.4396
		3.5	0.8606	0.0185	0.7310	4.2699
		5.0	0.8979	0.0097	0.5604	3.7904
		25	0.9778	0.0005	0.2661	3.2453
2.5	5.0	2.0	0.8451	0.0568	1.0909	5.9964
		3.5	0.8999	0.0202	0.6080	4.0605
		5.0	0.9265	0.0104	0.4375	3.6480
		25	0.9839	0.0005	0.1402	3.2214
5.0	5.0	2.0	0.9390	0.0670	0.9494	5.6318
		3.5	0.9561	0.0221	0.4675	3.9299
		5.0	0.9666	0.0110	0.2943	3.5912
		25	0.9923	0.0005	-0.0120	3.2971

The nth incomplete moment of the MOGFr distribution follows using Equation (6) as

$$\varphi_n(t) = \int_0^t x^n f(x)dx = \sum_{k=0}^{\infty} d_{k+1} \int_0^t x^n\, g_{\alpha(k+1)}(x)dx.$$

Then, we obtain

$$\varphi_n(t) = \sum_{k=0}^{\infty} d_{k+1}\, \alpha^n (k+1)^{n/\beta} \gamma\left(1-\frac{n}{\beta}, (k+1)\left(\frac{\alpha}{t}\right)^{\beta}\right), n < \beta. \quad (8)$$

Setting $n = 1$ in Equation (8), we obtain the first incomplete moment of X

$$\varphi_1(t) = \sum_{k=0}^{\infty} d_{k+1}\, \alpha (k+1)^{1/\beta} \gamma\left(1-\frac{1}{\beta}, (k+1)\left(\frac{\alpha}{t}\right)^{\beta}\right), \quad (9)$$

which has important applications related to the mean residual life, mean waiting time, Bonferroni and Lorenz curves.

3.3 Mean residual lifetime and mean waiting time

The mean residual life (MRL) (or life expectancy at age t) represents the expected additional life length for a unit, which is alive at age t and it is defined by $m_X(t) = E(X - t|X > t), t > 0$.

The MRL of X, can be defined as

$$m_X(t) = [1 - \varphi_1(t)]/S(t) - t, \quad (10)$$

where $S(t) = 1 - F(x)$ is the SF of the MOGFr distribution and $\varphi_1(t)$ is given in (9).

By substituting (9) in Equation (10), we have MRL of the MOGFr distribution as

$$m_X(t) = \frac{1}{S(t)} \sum_{k=0}^{\infty} d_{k+1}\, \alpha (k+1)^{1/\beta} \gamma\left(1-\frac{1}{\beta}, (k+1)\left(\frac{\alpha}{t}\right)^{\beta}\right) - t.$$

The mean waiting time (MWT) (or mean inactivity time) is defined by $M_X(t) = E[t - X|X \le t]$, $t > 0$, represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

The MWT of X can be defined as

$$M_X(t) = t - [\varphi_1(t)/F(t)]. \quad (11)$$

By inserting (9) in Equation (11), the MWT of the MOGFr distribution reduces to

$$M_X(t) = t - \frac{1}{F(t)} \sum_{k=0}^{\infty} d_{k+1}\, \alpha (k+1)^{1/\beta} \gamma\left(1-\frac{1}{\beta}, (k+1)\left(\frac{\alpha}{t}\right)^{\beta}\right).$$

3.4 Quantile and generating functions

The quantile function (QF) of X is obtained by inverting (4) as

$$x_u = \alpha \left\{-\frac{1}{\eta} \log\left(\frac{1-u}{1-(1-\delta)\, u}\right)\right\}^{\frac{-1}{\beta}}, 0 < u < 1. \quad (12)$$

The median of X follows by setting $u = 0.5$ in (12). The MOGFr random variable can be Simulated if U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = x_u$ at $u = U$ ollows Equation (5).

The moment generating function (MGF) of X follows from (6) as

$$M_X(t) = \sum_{k=0}^{\infty} d_{k+1}\, M_{k+1}(t),$$

where $M_{k+1}(t)$ is the MGF of the Fr distribution with parameters β and scale parameter $\alpha(k+1)^{1/\beta}$.

Afify et al. (2016b) provided a simple representation for the MGF, $M(t)$, of the Fr(α, β) distribution.

Consider the random variable Y ~Fr(α, β), and let $w = 1/y$, then the MGF of the Fr distribution reduces to

$$M(t) = \beta\alpha^{\beta} \int_0^{\infty} \exp(t/w) w^{\beta-1} \exp\left(-\alpha^{\beta} x^{-\beta}\right) dw.$$

Using the exponential series for $\exp(t/w)$ and after some simplifications, we have

$$M(t; \alpha, \beta) = \beta\alpha^{\beta} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} w^{\beta-n-1} \exp\left(-\alpha^{\beta} x^{-\beta}\right) dw$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n\, t^n}{n!}\, \Gamma\left(\frac{\beta-n}{\beta}\right).$$

The Wright generalized hypergeometric function is defined by

$${}_p\Psi_q \left[\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_p, A_p)\\ (\beta_1, B_1), \dots, (\beta_q, B_q)\end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j\, m)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j\, m)} \frac{x^m}{m!}.$$

Then, the MGF of the Fr model is

$$M(t; \alpha, \beta) =_1 \Psi_0 \left[\begin{matrix}(1, -\beta^{-1})\\ -\end{matrix}; \alpha\, t\right]. \qquad (13)$$

The MGF of the MOGFr distribution follows, by combining Equations (6) and (13), as

$$M(t) = \sum_{k=0}^{\infty} d_{k+1}\ {}_1\Psi_0 \left[\begin{matrix}(1, -\beta^{-1})\\ -\end{matrix}; \alpha\, (k+1)^{1/\beta}\, t\right].$$

3.5 Order statistics

Let $X_1, \dots, X_n$ be a random sample of size n from the MOGFr distribution and let $X_{1:n}, \dots, X_{n:n}$ be the corresponding order statistics. Then, the PDF of the of rth order statistic, $X_{r:n}$, is defined by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\, F(x)^{r-1} [1 - F(x)]^{n-r}.$$

Then, the PDF of the ith order statistic of the MOGFr distribution reduces to

$$f_{r:n}(x) = \frac{K\delta^{n-r+1}\eta\beta\alpha^{\beta}x^{-\beta-1}\exp(-\alpha^{\beta}x^{-\beta})}{\left\{1-(1-\delta)\left[1-\exp(-\alpha^{\beta}x^{-\beta})\right]^{\eta}\right\}^{n+1}}\left[1-\exp(-\alpha^{\beta}x^{-\beta})\right]^{\eta(n-r+1)-1}\left\{1-\left[1-\exp(-\alpha^{\beta}x^{-\beta})\right]^{\eta}\right\}^{r-1}, \quad (14)$$

where $K = n!/(r-1)!\,(n-r)!$.

Hence, the PDF of the first order statistic $X_{1:n}$ follows from (14) with $r = 1$, as

$$f_{1:n}(x) = \frac{n\delta^{n}\eta\beta\alpha^{\beta}x^{-\beta-1}\exp(-\alpha^{\beta}x^{-\beta})\left[1-\exp(-\alpha^{\beta}x^{-\beta})\right]^{\eta n-1}}{\left\{1-(1-\delta)\left[1-\exp(-\alpha^{\beta}x^{-\beta})\right]^{\eta}\right\}^{n+1}}.$$

The PDF of the largest order statistic $X_{n:n}$ is given by

$$f_{n:n}(x) = \frac{n\delta\eta\beta\alpha^{\beta}x^{-\beta-1}\exp(-\alpha^{\beta}x^{-\beta})\left[1-\exp(-\alpha^{\beta}x^{-\beta})\right]^{\eta-1}}{\left\{1-(1-\delta)\left[1-\exp(-\alpha^{\beta}x^{-\beta})\right]^{\eta}\right\}^{n+1}} \times\left\{1-\left[1-\exp(-\alpha^{\beta}x^{-\beta})\right]^{\eta}\right\}^{n-1}.$$

4. Maximum likelihood estimation

The estimation of the MOGFr parameters from complete samples only is considered by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a random sample of the MOGFr distribution with parameter vector $\theta = (\alpha, \beta, \delta, \eta)^{\intercal}$

The log-likelihood function for θ is

$$\ell = n\beta\log\alpha + n\log\beta + n\log\delta + n\log\eta - (\beta+1)\sum_{i=1}^{n}\log x_i - \alpha^{\beta}\sum_{i=1}^{n}x_i^{-\beta} + (\eta-1)\sum_{i=1}^{n}\log\left[1-\exp\left(-\alpha^{\beta}x_i^{-\beta}\right)\right] - 2\sum_{i=1}^{n}\log\left\{1-(1-\delta)\left[1-\exp\left(-\alpha^{\beta}x_i^{-\beta}\right)\right]^{\eta}\right\}. \quad (15)$$

The maximum likelihood estimators (MLEs) can be obtained by maximizing (15) either by using the different programs such as R, SAS or by solving the nonlinear likelihood equations obtained by differentiating (15).

The score vector elements, $\mathbf{U}(\Theta) = \frac{\partial\ell}{\partial\theta} = (\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial\eta})^{\intercal}$, are

$$\frac{\partial\ell}{\partial\alpha} = \frac{n\beta}{\alpha} + -\beta\alpha^{\beta-1}\sum_{i=1}^{n}x_i^{-\beta} + (\eta-1)\sum_{i=1}^{n}\frac{\beta\alpha^{\beta-1}x_i^{-\beta}\exp\left(-\alpha^{\beta}x_i^{-\beta}\right)}{1-\exp\left(-\alpha^{\beta}x_i^{-\beta}\right)} + 2\eta(1-\delta)\sum_{i=1}^{n}\frac{\beta\alpha^{\beta-1}x_i^{-\beta}\exp\left(-\alpha^{\beta}x_i^{-\beta}\right)\left[1-\exp\left(-\alpha^{\beta}x_i^{-\beta}\right)\right]^{\eta-1}}{1-(1-\delta)\left[1-\exp\left(-\alpha^{\beta}x_i^{-\beta}\right)\right]^{\eta}},$$

$$\frac{\partial \ell}{\partial \beta} = n\log\alpha + \frac{n}{\beta} - \sum_{i=1}^{n} \log x_i - \alpha^{\beta} \sum_{i=1}^{n} x_i^{-\beta} \log\left(\frac{\alpha}{x_i}\right)$$
$$+(\eta-1)\sum_{i=1}^{n} \frac{\alpha^{\beta} x_i^{-\beta} \exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\log\left(\frac{\alpha}{x_i}\right)}{1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)}$$
$$+\frac{2\eta(1-\delta)}{\alpha^{-\beta}}\sum_{i=1}^{n} \frac{\left[1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\right]^{\eta-1} x_i^{-\beta} \exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\log\left(\frac{\alpha}{x_i}\right)}{1-(1-\delta)\left[1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\right]^{\eta}},$$

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} - 2\sum_{i=1}^{n} \frac{\left[1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\right]^{\eta}}{1-(1-\delta)\left[1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\right]^{\eta}}$$

and

$$\frac{\partial \ell}{\partial \eta} = \frac{n}{\eta} + \sum_{i=1}^{n} \log\left[1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\right]$$
$$+2(1-\delta)\sum_{i=1}^{n} \frac{\left[1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\right]^{\eta}\log\left[1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\right]}{1-(1-\delta)\left[1-\exp\left(-\alpha^{\beta} x_i^{-\beta}\right)\right]^{\eta}}.$$

The estimates of the MOGFr parameters can be obtained by setting the score vector to zero, $\mathbf{U}(\hat{\theta}) = \mathbf{0}$ and solving these equations simultaneously gives the MLEs $\hat{\alpha}, \hat{\beta}, \hat{\delta}$ and $\hat{\eta}$. The interval estimation of the MOGFr parameters requires the 4×4 observed information matrix $J(\theta) = \{J_{rs}\}$ for $r, s = \alpha, \beta, \delta, \eta$. The approximate confidence intervals for the MOGFr parameters can be provided using the multivariate normal $N_4(0, J(\hat{\theta})^{-1})$ distribution, where $J(\hat{\theta})$ is the total observed information matrix evaluated at $\hat{\theta}$. Hence, we can determine the approximate $100(1-\varphi)\%$ confidence intervals for α, β, δ and η as follows:

$$\hat{\alpha} \pm z_{\varphi/2}\sqrt{\hat{J}_{\alpha\alpha}}, \quad \hat{\beta} \pm z_{\varphi/2}\sqrt{\hat{J}_{\beta\beta}}, \quad \hat{\delta} \pm z_{\varphi/2}\sqrt{\hat{J}_{\delta\delta}} \quad \text{and} \quad \hat{\eta} \pm z_{\varphi/2}\sqrt{\hat{J}_{\eta\eta}},$$

where $z_{\varphi/2}$ is the upper φth percentile of the standard normal distribution.

5. Two applications

In this section, we ilustrate the flexibility and importance of the MOGFr distribution empirically by two real data applications. The first data set contains 101 observations with maximum stress per cycle 31,000 psi. The data refer to the fatigue life of 6061-T6 aluminum coupons (Birnbaum and Saunders, 1969). The second data set consists of 128 observations of bladder cancer patients which represents the remission times (in months) (Lee and Wang, 2003). Table 3 lists the competitive models of the MOGFr distribution which will be compared with it.

Table 3: Fitted competitive distributions of the MOGFr model

Distribution	Author(s)
Fréchet (special case of MOGFr) (Fr)	Fréchet (1924)
Transmuted exponentiated Fr (TEFr)	Elbatal et al. (2014)
Weibull Fr (WFr)	Afify et al. (2016b)
Kumaraswamy Fr (KFr)	Mead and Abd-Eltawab (2014)
Marshall-Olkin Fr (MOFr)	Krishna et al. (2013)
Burr X Fr (BXFr)	Abouelmagd et al. (2018)
Exponentiated Fr (EFr)	Nadarajah and Kotz (2003)
Modified Fr (MFr)	Tablada and Cordeiro (2017)

We shall consider the minus log-likelihood $(-\hat{\ell})$, Kolmogorov Smirnov (KS) statistic, its P-value (PV), Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics to compare the fitted distributions.

Tables 4 and 5 list the values of the MLEs and their corresponding standard errors (in parentheses) of the MOGFr parameters and other fitted models parameters. These tables also show the values $-\hat{\ell}$, KS, PV, W^* and A^* statistics for both data sets.

In Tables 4 and 5, we compare the MOGFr model with the TEFr, WFr, KFr, MOFr, BXFr, EFr, MFr and Fr distributions. We note that the MOGFr model gives the lowest values for the $-\hat{\ell}$, KS, W^* and A^* statistics and the largest value of the PV among all fitted models. Hence, the MOGFr model could be chosen as the best model to explain both data sets.

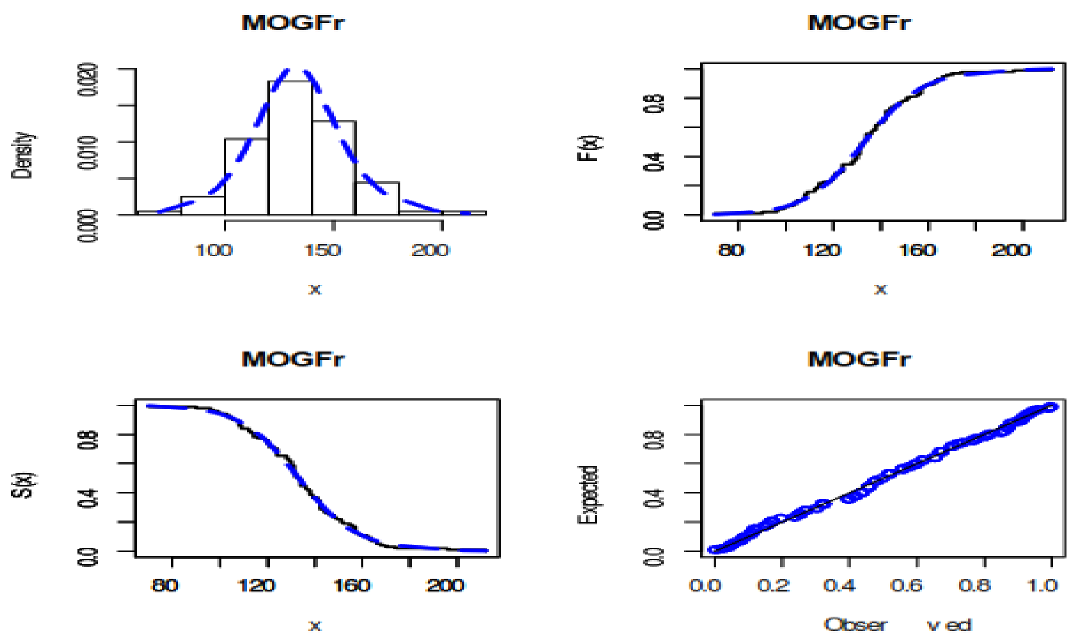

Figure 2: Fitted PDF, CDF, SF and PP plots of the MOGFr distribution for fatigue life data

The histogram of both data sets, and the estimated CDF, SF and PP plots for the MOGFr distribution are shown in Figures 2 and 3.

Table 4: The MLEs, their (SEs) and the $-\hat{\ell}$, KS, PV, W^* and A^* measures for fatigue life data

Model	Estimates	$-\hat{\ell}$	KS	PV	W^*	A^*
MOGFr $(\alpha, \beta, \delta, \eta)$	108.902, 2.8169, 70.3445, 5.1022 (58.3923, 1.8775, 235.69, 5.6138)	455.193	0.0503	0.9601	0.0366	0.2536
TEFr (α, β, a, b)	1055.2, 0.9667, 627.01, 0.8162 (1994.1, 0.5936, 2767.8, 0.1940)	455.756	0.0592	0.8703	0.0470	0.3036
WFr (α, β, η, b)	618.600, 1.5242, 63.6567, 0.4329 (802.57, 0.4336, 77.0615, 0.8328)	456.170	0.0678	0.7415	0.0544	0.3417
KFr (α, β, a, b)	112.60, 1.5159, 5.7744, 61.705 (6475.7, 0.4778, 503.41, 86.662)	456.254	0.0670	0.7549	0.0548	0.3475
MOFr (α, β, θ)	63.101, 10.661, 2738.8 (6.0324, 0.8804, 2095.1)	455.739	0.0616	0.8380	0.0606	0.3681
BXFr (α, β, θ)	88.129, 1.3818, 3.4810 (18.433, 0.3542, 2.3911)	456.248	0.0683	0.7333	0.0574	0.3589
EFr (α, β, θ)	357.99, 1.5159, 61.699 (185.95, 0.4775, 86.602)	456.254	0.0670	0.7549	0.0548	0.3475
MFr (α, β, θ)	36510, 0.8303, 0.0386 (14462, 0.0724, 0.0027)	463.982	0.1121	0.1579	0.2261	1.2757
Fr (α, β)	120.78, 5.0574 (2.5251, 0.3252)	475.185	0.1329	0.0563	0.4330	2.4970

The plots in these figures show that the MOGFr distribution has a close fits to both data sets.

Figure 4 displays the HRF plots of the MOGFr distribution for both data sets. It is seen that, the HRF is increasing for the fatigue life data and upside down bathtub for the cancer data.

Table 5: The MLEs, their (SEs) and the $-\hat{\ell}$, KS, PV, W^* and A^* measures for cancer data

Model	Estimates	$-\hat{\ell}$	KS	PV	W^*	A^*
MOGFr $(\alpha,\beta,\delta,\eta)$	10.2062, 0.2993, 62.4976, 10.9617 (35.6278, 0.1940, 139.206,13.4102)	409.370	0.0297	0.9998	0.0150	0.0932
TEFr (α,β,a,b)	2806.9, 0.2251, 56.336, -0.7044 (4498.9, 0.0355, 35.687, 0.2143)	410.486	0.0366	0.9955	0.0336	0.2374
WFr (α,β,η,b)	118.595, 0.2088, 36.738, 2.3771 (389.26, 0.0773, 88.5041, 1.1208)	411.511	0.0546	0.8388	0.0634	0.4109
KFr (α,β,a,b)	0.8244, 0.1635, 9.4232, 596.51 (17.890, 0.0546, 33.631, 1310.8)	411.117	0.0505	0.8995	0.0524	0.3468
MOFr (α,β,θ)	0.0475, 1.7248, 4315.3 (0.0185, 0.1257, 1076.0)	411.457	0.0399	0.9869	0.0441	0.3189
BXFr (α,β,θ)	1.4349, 0.2867, 1.8324 (1.1631, 0.0622, 0.9529)	411.447	0.0509	0.8951	0.0556	0.3771
EFr (α,β,θ)	33212, 0.1998, 180.42 (3444.4, 0.0053, 39.489)	411.339	0.0484	0.9253	0.0522	0.3556
MFr (α,β,θ)	16.474, 0.3807, 0.1055 (9.3401, 0.0703, 0.0174)	413.524	0.0649	0.6543	0.1192	0.7335
Fr (α,β)	3.2582, 0.7520 (0.4074, 0.0424)	444.001	0.1408	0.0125	0.7443	4.5464

Figure 5 shows the TTT plots of the MOGFR distribution for both data sets. The TTT plot for fatigue life data is concave which indicates that it has an increasing hazard rate, whereas the TTT plot for the cancer data is concave then convex which indicates an upside down bathtub hazard rate. Hence, the MOGFr distribution is a suitable for modeling both data sets.

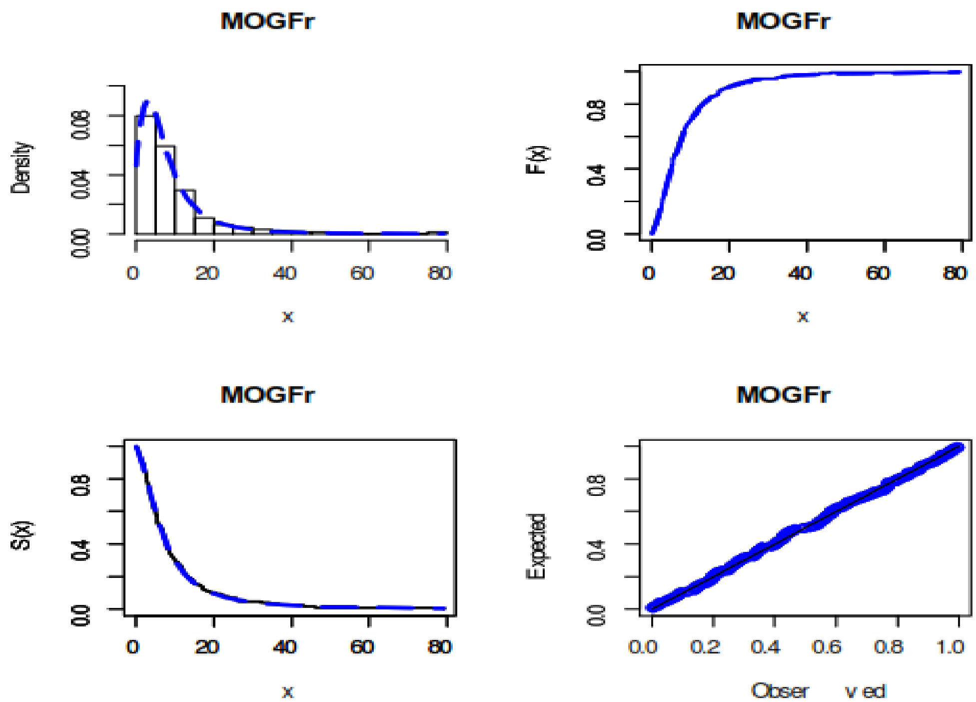

Figure 3: Fitted PDF, CDF, SF and PP plots of the MOGFr distribution for cancer data

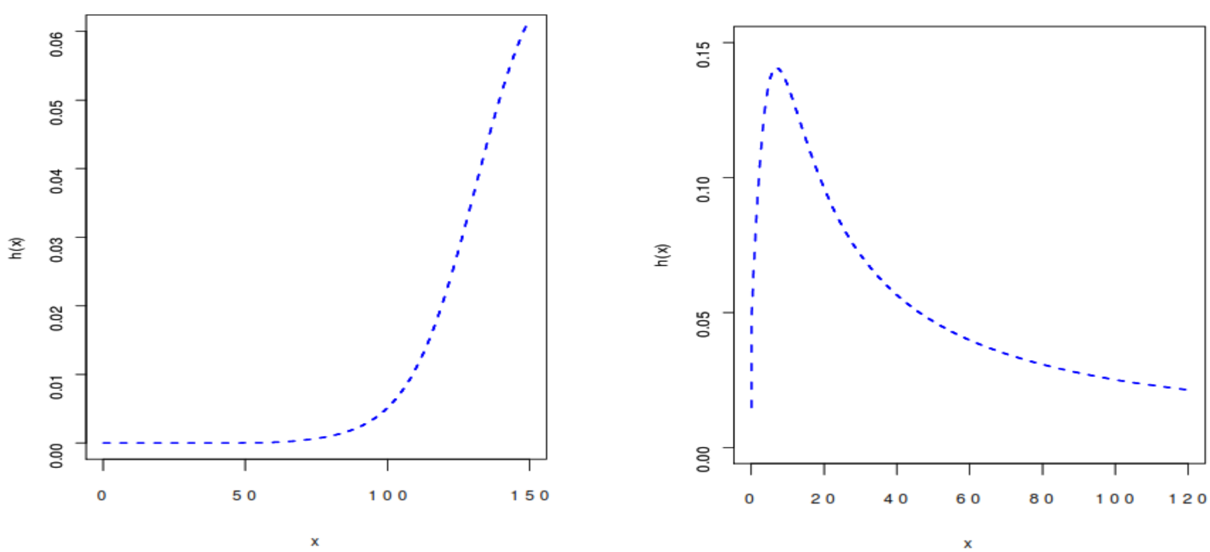

Figure 4: The HRF plots of the MOGFr distribution for fatigue life data (left panel) and for cancer data (right panel)

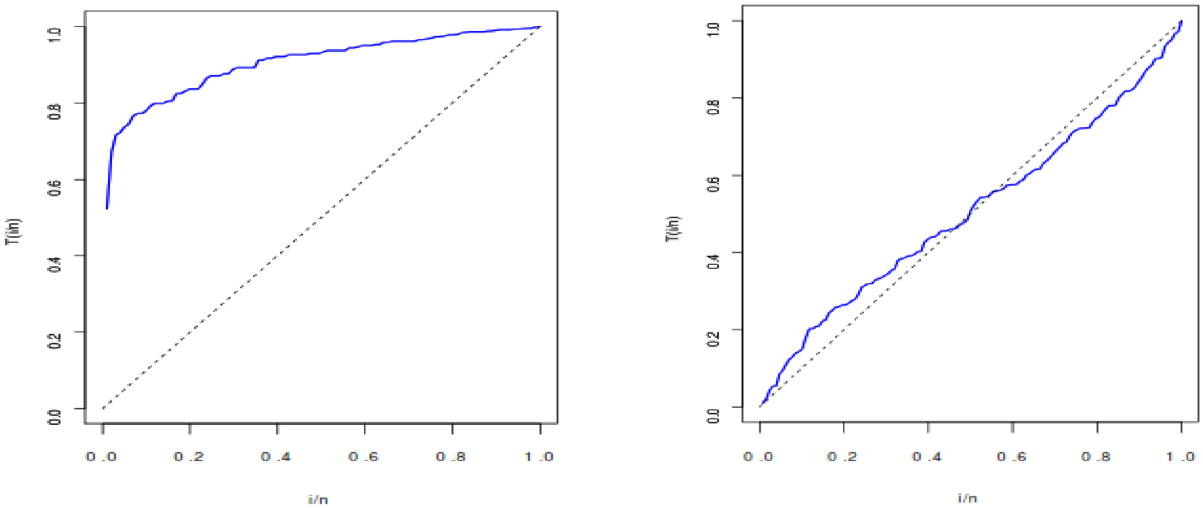

Figure 5: The TTT plots of the MOGFr distribution for fatigue life data (left panel) and for cancer data (right panel)

6. Conclusions

In this paper, we propose a new four-parameter model called the MarshallOlkin generalized Fréchet (MOGFr) distribution, which contains the Fréchet, Marshall-Olkin Fréchet and exponentiated Fréchet distributions, among others as special cases. The MOGFr density function can be expressed as a linear mixture of Fréchet densities. Explicit expressions for some of its mathematical quantities including the quantile and generating functions ordinary and incomplete moments, mean residual life, mean waiting time and order statistics are derived. The MOGFr parameters are estimated by the maximum likelihood method. The proposed distribution provides better fits than some other nested and non-nested models by using two real data sets.

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